TANGENTIALLY CUBIC SUBMANIFOLDS OF $\mathbb{E}^m$

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Abstract. In the present study we consider the submanifold $M$ of $\mathbb{E}^m$ satisfying the condition $\langle \Delta H, e_i \rangle = 0$, where $H$ is the mean curvature of $M$ and $e_i \in TM$. We call such submanifolds tangentially cubic. We proved that every null 2-type submanifold $M$ of $\mathbb{E}^m$ is tangentially cubic. Further, we prove that the pointed helical geodesic surfaces of $\mathbb{E}^5$ with constant Gaussian curvature are tangentially cubic.

1. Introduction

Let $x : M \rightarrow \mathbb{E}^m$ be an isometric immersion from an $n$-dimensional connected manifold $M$ into the Euclidean $m$-space $\mathbb{E}^m$. With respect to the Riemannian metric $g$ on $M$ induced from the Euclidean metric of the ambient space $\mathbb{E}^m$, $M$ is a Riemannian manifold $(M, g)$. Denote by $\Delta$ the Laplacian operator of the Riemannian manifold $(M, g)$. One of the most important formulas in Differential Geometry of submanifolds is

$$\Delta x = -nH,$$

where $H$ is the mean curvature vector field of the immersion, and $x$ also denotes the position vector field of $M$ in $\mathbb{E}^m$. Formula (1.1) implies that the immersion is minimal ($H = 0$) if and only if the immersion is harmonic, that is $\Delta x = 0$. An isometric immersion $x : M \rightarrow \mathbb{E}^m$ is called biharmonic if we have $\Delta^2 x = 0$, that is $\Delta H = 0$. It is obvious that minimal immersions are biharmonic [3].

$M$ is said to be of null 2-type submanifold of $\mathbb{E}^m$ if each component of the position vector $x$ has a finite spectral decomposition (see, [4])

$$x = x_0 + x_1, \quad \Delta x_0 = 0, \quad \Delta x_1 = cx_1,$$

for some non-constant vectors $x_0$ and $x_1$ on $M$, where $c$ is a non-zero constant.

In [2] the present authors considered the differentiable curve $\gamma$ in $\mathbb{E}^m$ satisfying the relation $\langle \Delta H, \gamma' \rangle = 0$. Such curves are called tangentially cubic, where $H$ is the mean curvature vector of $\gamma$.

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In the present study we extend the results in [2] to the submanifolds of $\mathbb{E}^m$. The submanifolds satisfying the condition
\begin{equation}
(\Delta H, e_i) = 0, \quad 1 \leq i \leq n, \quad e_i \in TM
\end{equation}
are called tangentially cubic (T.C - submanifolds). We show that the hypercylinder over the tangentially cubic curves is also tangentially cubic. Further, we give some examples of T.C-submanifolds.

In [6] Y.H. Kim studied the submanifolds which has pointed helical geodesics with the same constant Frenet curvatures. We prove that the helical geodesics of $\mathbb{E}^5$ with constant Gaussian curvature are tangentially cubic surfaces.

2. Basic Concepts

Let $x : M \rightarrow \mathbb{E}^m$ be an isometric immersion from an $n$-dimensional, connected manifold $M$ into the Euclidean $m$-space $\mathbb{E}^m$. Let $\nabla$ and $\nabla_\parallel$ denote the covariant derivatives of $M$ and $\mathbb{E}^m$ respectively. Thus $\nabla_\parallel X$ is just the directional derivative in the direction $X$ in $\mathbb{E}^m$.

Then for tangent vector fields $X, Y$ the second fundamental form $h$ of the immersion is defined by
\begin{equation}
h(X, Y) = \nabla_\parallel X Y - \nabla X Y.
\end{equation}
For a vector field $\xi$ normal to $M$ we put
\begin{equation}
\nabla_\parallel X \xi = -A_X \xi + D_X \xi,
\end{equation}
where $-A_X \xi$ (resp. $D_X \xi$) denotes the tangential and normal component of $\nabla_\parallel X \xi$ and $D$ is the normal connection of $M$.

Let us choose a local field of orthonormal frame $\{e_1, e_2, ..., e_n, e_{n+1}, ..., e_m\}$ in $\mathbb{E}^m$ such that, restricted to $M$, the vectors $e_1, e_2, ..., e_n$ tangent to $M$ and $e_{n+1}, ..., e_m$ are normal to $M$. We denote by $\{w^1, w^2, ..., w^m\}$ the field of dual frames. The structure equations of $\mathbb{E}^m$ are given by (see [3])
\begin{equation}
\nabla_{e_i} e_j = \sum_{k=1}^{n} w^k_j (e_i) e_k + \sum_{\alpha=n+1}^{m} w^\alpha_j (e_i) e_\alpha.
\end{equation}

The mean curvature vector of $M$ is
\begin{equation}
H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i).
\end{equation}

If $H = 0$, then $M$ is said to be minimal.

The Laplace operator $\Delta$ acting on a vector valued function $V$ is given by
\begin{equation}
\Delta V = \sum_{i=1}^{n} \left[ \nabla \nabla_{e_i} V - \nabla_{e_i} \nabla_{e_i} V \right].
\end{equation}

We define the Laplacian $\Delta^D$ with respect to the normal connection $D$
\begin{equation}
\Delta^D H = \sum_{i=1}^{n} \left[ D_{\nabla_{e_i} e_i} H - D_{e_i} D_{e_i} H \right].
\end{equation}

3. Main Results

Let $M$ be a $H$-hypersurface in $\mathbb{E}^{n+1}$ then applying (2.3) to $H$, since $H = \alpha N$, we find
\begin{equation}
\Delta H = 2A_N \text{grad} \alpha + n \text{grad} \alpha + (\Delta \alpha + S \alpha) N,
\end{equation}
where $\alpha$ and $S$ stand for the mean curvature and the square of the length of the second fundamental form, respectively. Suppose that the hypersurface $M$ in the Euclidean space $\mathbb{E}^{n+1}$ is biharmonic. Then from (3.1) we have

\begin{equation}
2\text{Agrad} \alpha + n \text{agrad} \alpha = 0
\end{equation}

and

\begin{equation}
\Delta \alpha + S \alpha = 0.
\end{equation}

The relations (3.2) and (3.3) are necessary and sufficient conditions for $M$ to be biharmonic. The hypersurfaces which satisfy (3.2) are called $H$-hypersurfaces [5].

First we prove the following result.

**Proposition 3.1.** Every $H$-hypersurface is a trivial T.C. hypersurface.

**Proof.** Let $M$ be a $H$-hypersurface in $\mathbb{E}^{n+1}$ then using (3.2) with (3.1) we get

\begin{equation}
\Delta H = (\Delta \alpha + S \alpha)N.
\end{equation}

So by the use of (3.4) we get

\[ \langle \Delta H, e_i \rangle = 0, \]

which completes the proof. $\square$

**Proposition 3.2.** Every biharmonic submanifold of $\mathbb{E}^m$ is trivial T.C. submanifold.

**Proof.** Let $M$ be an $n$-dimensional connected submanifold of $\mathbb{E}^m$. Then by the Beltrami formula (1.1) we get

\begin{equation}
\langle \Delta H, e_i \rangle = -\frac{1}{n} < \Delta^2 x, e_i >, \quad 1 \leq i \leq n,
\end{equation}

which completes the proof. $\square$

**Lemma 3.1.** [4] Let $M$ be an $n$-dimensional submanifold of an Euclidean space $\mathbb{E}^m$. If there is a constant $c \neq 0$ such that $\Delta H = c H$, then $M$ is either of 1-type or of null 2-type.

**Proposition 3.3.** [3] Let $M$ be an $n$-dimensional submanifold of an $m$-dimensional Riemannian manifold $\mathbb{E}^m$. Let $e_{n+1}, \ldots, e_m$ be mutually orthogonal unit normal vector fields of $M$ in $\mathbb{E}^m$ such that $e_{n+1}$ is parallel to the mean curvature vector $H$ of $M$ in $\mathbb{E}^m$ then

\begin{equation}
\Delta H = \Delta^P H + \|A_{n+1}\|^2 H + a(H) + \text{tr}(\tilde{\nabla} A_H),
\end{equation}

where

\begin{equation}
a(H) = \sum_{r=n+2}^{m} \text{tr}(A_H A_r)e_r, \quad A_r = A_{e_r}, \quad n + 2 \leq r \leq m,
\end{equation}

and

\begin{equation}\|A_{n+1}\|^2 = \text{tr}(A_{n+1}A_{n+1}),\end{equation}

\begin{equation}\text{tr}(\tilde{\nabla} A_H) = \sum_{i=1}^{n} [\langle \nabla_{e_i} A_H \rangle e_i + A_{D_{e_i} H} e_i].\end{equation}
Consider the case when
\[ \text{Let } \]

If
\[ \text{Let } \]

the submanifolds
\[ \text{Theorem 3.2.} \]

tangent frame of the cylinder by
\[ \text{isometric immersions of closed manifolds and } \]

\[ \text{is called the product immersion of } \]
\[ f \]

\[ \text{function } \]
\[ f \]

\[ \text{Proof. } \]
\[ \text{Let } \]

\[ \text{Lemma 3.2.} \]

\[ \text{tr}(\nabla A_H) = 0 \]

and
\[ \Delta H = \Delta^D H + \|A_{n+1}\|^2 H + a(H). \]

Consequently we have the following result.

**Proposition 3.4.** Let \( M \) be an \( n \)-dimensional submanifold of an Euclidean space \( \mathbb{E}^m \). If \( M \) is of null 2-type (i.e. not of 1-type) then \( M \) is a T.C-submanifold.

**Proof.** If \( M \) is of null 2-type then (3.9) and (3.10) are full filled. So using (3.10) we get
\[ < \Delta H, e_1 > = 0, \]
which completes the proof. \( \square \)

**Definition 3.1.** Consider the case when \( M = M_1 \times M_2 \) is a product submanifold. That is, there exist isometric embeddings
\[ \text{(3.11) } \]

\[ f_1 : M_1 \to \mathbb{E}^{m_1+d_1}, \quad f_2 : M_2 \to \mathbb{E}^{m_2+d_2}. \]

We put \( m = m_1 + m_2, d = d_1 + d_2 \) so that \( \mathbb{E}^{m+d} = \mathbb{E}^{m_1+d_1} + \mathbb{E}^{m_2+d_2} \). Then the function \( f(x_1, x_2) = (f_1(x_1), f_2(x_2)) \) defines an embedding \( f : M \to \mathbb{E}^{m+d} \) which is called the product immersion of \( f_1, f_2 \) (see, [7]).

**Theorem 3.1.** [1] Let \( f_1 : M_1 \to \mathbb{E}^{m_1+d_1} \) and \( f_2 : M_2 \to \mathbb{E}^{m_2+d_2} \) be two isometric immersions of closed manifolds and \( \Delta, \Delta_1 \) and \( \Delta_2 \) be the Laplacian of the submanifolds \( M = M_1 \times M_2 \), \( M_1 \) and \( M_2 \) respectively. Then
\[ \Delta = \Delta_1 + \Delta_2. \]

**Theorem 3.2.** Let \( \gamma \) be a differentiable curve in \( \mathbb{E}^m \). If \( \gamma \) is a T.C-curve then the cylinder over \( \gamma \) is also a T.C-surface.

**Proof.** Let \( \gamma(s) = (\gamma_1(s), \gamma_2(s), ..., \gamma_n(s)) \) be the curve in \( \mathbb{E}^m \). The cylinder over \( \gamma \) will have the parametrization
\[ x = (s, u_1, u_2, ..., u_{n-1}) = (\gamma(s), u_1, u_2, ..., u_{n-1}). \]

Let \( \gamma'(s) = v_1, v_2, ..., v_n \) be the oriented frame field of \( \gamma \). We chose an orthonormal tangent frame of the cylinder by \( \{x_s, x_{u_1}, x_{u_2}, ..., x_{u_{n-1}}\} \), where
\[ x_s = (v_1, 0, ..., 0) \]
\[ x_{u_j} = (0, 0, ..., 1, ..., 0), \quad 1 \leq j \leq n-1. \]

A simple calculation gives
\[ \nabla_{x_s} x_s = 0, \quad \nabla_{x_s} x_{u_j} = 0 = \nabla_{x_{u_j}} x_s = 0, \quad \nabla_{x_{u_j}} x_{u_k} = 0 \]
and
\[ h(x_s, x_s) = (\gamma_1''(s), \gamma_2''(s), ..., \gamma_n''(s), 0, 0, ..., 0), \]
\[ h(x_s, x_{u_j}) = h(x_{u_j}, x_{u_k}) = 0. \]
So the mean curvature vector of the cylinder will become

\[ H = \frac{1}{n} \sum_{i=1}^{n-1} \left( h(x_s, x_s) + h(x_{u_i}, x_{u_i}) \right) \]

which is equal to the second derivative of \( \gamma \) with \( n - 1 \) zeros will be added. If \( \gamma \) is a T.C.-curve then the cylinder \( \gamma \times E^{n-1} \) will be a T.C.-surface.

We give the following examples.

**Example 3.1.** The helix in \( S^3 \subset E^4 \) given by the parametrization

\[ \gamma(s) = (\cos \phi \cos(as), \cos \phi \sin(as), \sin \phi \cos(bs), \sin \phi \sin(bs)) \]

is a T.C.-curve in \( S^3 \subset E^4 \)(see, [2]). Hence, the cylinder \( M \) over \( \gamma \) given with the parametrization

\[ x(s, t) = (\cos \phi \cos(as), \cos \phi \sin(as), \sin \phi \cos(bs), \sin \phi \sin(bs), t) \]

is a T.C.-surface.

**Example 3.2.** The product manifold of Catenoid with the circle \( S^1(b) \) is given by

\[ x(s, u_1, u_2) = (b \cos s, b \sin s, a \cosh u_1 \cos u_2, a \cosh u_1 \sin u_2, au_1) \]

In [1] it has been shown that the product immersion \( x(s, u_1, u_2) \) is of null 2-type. So, by Theorem 3.2 the product submanifold given with the parametrization (3.14) is a T.C.-submanifold.

In [6] Y.H. Kim studied the submanifolds which has pointed helical geodesics with the same constant Frenet curvatures. He proved the following result.

**Proposition 3.5.** [6] Let \( M \subset E^5 \) be a compact connected surface fully lies in \( E^5 \). If \( M \) has pointed helical geodesics with the same constant Frenet curvatures then it has the parametrization

\[ x(s, \theta) = \left( \frac{1}{k} \sin ks \cos \theta, \frac{1}{k} \sin ks \sin \theta, \frac{1}{k^2} (1 - \cos ks) \left( k - \frac{2a^2}{k} \sin^2 \theta \right) \right), \]

where \( k \) is the Frenet curvature of the helical geodesic on \( M \) and

\[ a = \|h(e_1, e_2)\|, b^2 = k^2 - \frac{(k^2 - 2a^2)^2}{k^2}. \]

**Proposition 3.6.** Let \( M \subset E^5 \) be a compact connected surface fully lies in \( E^5 \). If \( M \) has pointed helical geodesics with the same constant Frenet curvatures and has constant Gaussian curvature then it is a T.C.-surface.

**Proof.** Let \( M \) be a proper surface of \( E^5 \). If \( M \) has pointed helical geodesics with the same constant Frenet curvatures then by Proposition 3.5 it has the parametrization of the form (3.15). Further, we assume that the Gaussian curvature of \( M \) is constant. So by Lemma 2.14 of [6] the Laplacian operator \( \Delta \) of \( M \) is given by

\[ \Delta = -\left( \frac{\partial^2}{\partial s^2} + \frac{1}{G} \frac{\partial^2}{\partial \theta^2} \right) - \frac{1}{2} \frac{\partial}{\partial s} \left( \log G \right) \frac{\partial}{\partial s} \]
where
\[ G = \frac{1}{k^2} \sin^2 ks + \frac{1}{k^2} (1 - \cos ks)^2. \]
Using Beltrami formula (1.1) and computing \( H \) where the means of (3.16), we obtain the following
\[ \Delta H - \frac{3}{2}k^2 H = 0. \]
So, by the use of Lemma 3.1 and Proposition 3.4 \( M \) becomes a \( T.C \)-surface. \( \square \)

**References**


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